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ALTERNATING RENEWAL PROCESS WITH EXPONENTIALLY DISTRIBUTED TIME TO FAILURE. BOUNDS WITH PRECISION.

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Naval Postgraduate School

THESIS

ALTERNATING RENEWAL PROCESS WITH EXPONENTIALLY DISTRIBUTED TIME TO FAILURE. BOUNDS WITH PRECISION.

by

Tor Mikal Nikolaisen

Thesis Advisor: R. W. Butterworth

September 1971

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Alternating Renewal Process with Exponentially Distributed Time to Failure. Bounds with Precision.

by

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Submitted in partial fulfillment of the
requirements for the degree of

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from the

NAVAL POSTGRADUATE SCHOOL
September 1971

ABSTRACT

This paper considers an alternating renewal process with time to failure being distributed exponential (λ), and with a general distribution for time to repair. Bounds and the precision of the bounds for the availability function were obtained.

A computer program was written to solve for the availability function and some other quantities when the down distribution was a gamma (α,β), α integer. The availability function with down distribution being a mixture of gamma-functions is considered.

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I. DESCRIPTION OF ALTERNATING RENEWAL PROCESS

A. SYSTEM

The system considered consists of a single unit or component which is repaired upon failure and then returned to operation.

In the model, the following assumptions are made:

- Time to failure is a random variable (R.V.) X with distribution F (specified below).

- Time to perform a repair is a R.V. Y with distribution G (specified below).

- Repair commences immediately upon failure.

- Unit is returned to operating state immediately upon completion of repair.

- Unit is as good as new after repair.

- Consecutive operating times between failures (as well as repair times) are independent.

Thus the system can be modeled as an alternating renewal process.

B. DISTRIBUTIONS

The time to failure - termed uptime - is throughout assumed to be exponentially distributed, with probability density function (p.d.f.):

$$f(t) = \begin{cases} \lambda e^{-\lambda t} , & t \geq 0 \\ 0 , & t < 0 \\ \lambda > 0 . & \end{cases} \tag{1.1}$$

Three different distributions for time to repair - termed downtime - were treated:

- Gamma distribution.
- Mixture of two exponential distributions.
- Mixture of gamma and exponential distribution.

The respective p.d.f.'s are:

$$(1.2)\qquad g(t) = \begin{cases} \dfrac{\beta^{\alpha}}{(\alpha-1)!}\, t^{\alpha-1}\, e^{-\beta t} & ; \quad t \geq 0 \\ 0 & ; \quad t < 0 \\ \beta > 0 & \\ \alpha = 1,2,3,\cdots & \end{cases}$$

$$(1.3)\qquad g(t) = \begin{cases} m\beta_1 e^{-\beta_1 t} + (1-m)\beta_2 e^{-\beta_2 t} & ; \quad t \geq 0 \\ 0 & ; \quad t < 0 \\ \beta_1, \beta_2 > 0 & \\ 0 \leq m \leq 1. & \end{cases}$$

$$(1.4)\qquad g(t) = \begin{cases} (1-m)\beta_1 e^{-\beta_1 t} + m\beta_2^2 t e^{-\beta_2 t} & ; \quad t \geq 0 \\ 0 & ; \quad t < 0 \\ \beta_1, \beta_2 > 0 & \\ 0 \leq m \leq 1. & \end{cases}$$

C. LAPLACE TRANSFORMS; GENERAL

Denoting the operating state by 1 and the failed state by 0, the objective is to find:

- $P_{11}(t)$: The "availability" of the system, i.e., the probability that the system will be "up" or operable

when called upon at a random time t, given that it starts in state 1 at t = 0.

The following are also of interest:

- $M_{11}(t)$: the expected number of visits to state 1 in period (0,t), given that the system is in state 1 at t = 0.

- $P_{01}(t)$: probability system is in state 1 at time t, given it starts in state 0 at t = 0.

- $E_{11}(T)$: expected time system is "on" in (0,T).

It is shown in [1] that

(1.5)

$$M_{11}(t) = \int_0^t M_{01}(t-x)\,dF(x)$$

(1.6)

$$M_{01}(t) = \int_0^t [1 + M_{11}(t-x)]\,dG(x).$$

Taking Laplace transform on both sides of (1.5) and (1.6), denoting the transform by circumflex ("hat"), i.e., $\hat{M}_{ij}(s) = \int_0^\infty e^{-st} M_{ij}(t)dt$, one obtains:

(1.7)

$$\hat{M}_{11}(s) = \frac{\hat{f}(s)\cdot\hat{g}(s)}{s(1-\hat{f}(s)\cdot\hat{g}(s))} .$$

In similar manner:

(1.8)

$$\hat{P}_{11}(s) = \frac{1-\hat{f}(s)}{s(1-\hat{f}(s)\cdot\hat{g}(s))} .$$

(1.9)

$$\hat{P}_{01}(s) = \frac{(1-\hat{f}(s))\ \hat{g}(s)}{s(1-\hat{f}(s)\cdot\hat{g}(s))} \quad .$$

(1.10)

$$\hat{E}(s) = \frac{1-\hat{f}(s)}{s^2(1-\hat{f}(s)\hat{g}(s))} \quad .$$

Only for some of the distributions considered can the inversion be accomplished in a closed form. For the general case with gamma down distribution, a computer program was written to give numerical solutions.

If a digital computer is not available, good approximations are of interest. These - together with bounds - have been developed for the availability function.

Some solutions in closed form in the simpler cases are provided.

II. EXACT SOLUTION - LAPLACE TRANSFORM

Laplace transforms of the up-and downtime probability density functions considered:

(2.1)

$$\hat{f}(s) = \frac{\lambda}{\lambda+s} \qquad \text{Ref. (1.1)}$$

(2.2)

$$\hat{g}(s) = \left(\frac{\beta}{\beta+s}\right)^{\alpha} \qquad \text{Ref. (1.2)}$$

(2.3)

$$\hat{g}(s) = m\,\frac{\beta_1}{\beta_1+s} + \left(1-m\right)\frac{\beta_2}{\beta_2+s} \qquad \text{Ref. (1.3)}$$

(2.4)

$$\hat{g}(s) = m\,\frac{\beta_1}{\beta_1+s} + \left(1-m\right)\left(\frac{\beta_2}{\beta_2+s}\right)^2 . \qquad \text{Ref. (1.4)}$$

In the case of (2.2) one obtains upon substituting into (1.7) to (1.10) and rearranging:

(2.5)

$$\hat{M}_{11}(s) = \frac{\lambda\beta^{\alpha}}{s\cdot\hat{q}(s)}$$

where,

(2.6)

$$\hat{q}(s) = (s+\beta)^{\alpha} + \lambda\sum_{i=1}^{\alpha} \beta^{i-1}(s+\beta)^{\alpha-i} .$$

(2.7)

$$\hat{P}_{11}(s) = \frac{(s+\beta)^{\alpha}}{s\cdot\hat{q}(s)} .$$

(2.8)

$$\hat{P}_{01}(s) = \frac{\beta^{\alpha}}{s\cdot\hat{q}(s)} .$$

(2.9)

$$\hat{E}_{11}(s) = \frac{(s+\beta)^{\alpha}}{s^2\hat{q}(s)} .$$

In cases (2.3) and (2.4), only $\hat{P}_{11}(s)$ has been considered:

(2.10)

$$\hat{P}_{11}(s) = \frac{(s+\beta_1)(s+\beta_2)}{s[s^2+(\lambda+\beta_1+\beta_2)s+\beta_1(\lambda+\beta_2)-\lambda m(\beta_1-\beta_2)]} \quad ; \text{ Ref. } (2.3)$$

(2.11)

$$\hat{P}_{11}(s) = \frac{(s+\beta_1)\ (s+\beta_2)^2}{s[s^3+As^2+Bs+C]} \quad ; \text{ Ref. } (2.4)$$

where

$$A = \beta_1 + 2\beta_2 + \lambda$$

$$B = \beta_2^2 + 2\beta_2(\lambda+\beta_1) + \lambda\beta_1 m$$

$$C = \lambda\beta^2(1-m) + \beta_1\beta_2^2 + 2\lambda\beta_1\beta_2 m.$$

III. INVERSION

A. COMMENTS

Consider the transform of a rational function $\hat{P}(s)/s\hat{q}(s)$, where the degree of $\hat{P}(s)$ is less than $s\hat{q}(s)$. This is a sufficient condition for the use of the residue theorem in finding the inverse Laplace transform [2], thus:

(3.1)

$$F(t) = \Sigma \text{ (Residues of } e^{st} f(s) \text{ at poles of } \hat{F}(s)).$$

As is shown in Appendix B only simple poles exist if $\beta > \lambda$. This simplifies the inversion.

Equation (2.9) has a double pole at $s = 0$. The appropriate adjustment to the procedure used is considered later.

In the sequel, primarily the inversion of $\hat{P}_{11}(s)$ will be considered. However, since the method used is applicable to all cases where Laplace transform is of type (3.1), the discussion in some detail of inversion of $\hat{P}_{11}(s)$ will show how the other results were obtained.

B. INVERSION FORMULA

Referring back to (2.7)

(3.2)

$$P_{11}(t) = L^{-1}\left(\frac{(s+\beta)^{\alpha}}{s\hat{q}(s)}\right)$$

$\hat{q}(s)$ is a polynomial of degree α. It is shown in Appendix A that all roots are complex, except when α is odd, in which case one of the roots is real and negative.

Factoring $\hat{sq}(s)$ into linear terms:

(3.3)

$$\hat{sq}(s) = s(s-r_1)(s-r_2)\cdots(s-r_\alpha)$$

where

$$r_j = a_j + b_j i \qquad (b_1 = 0 \text{ if } \alpha \text{ is odd})$$

$$r_{j+1} = a_j - b_j i \qquad j = 1,2,\cdots,\alpha-1.$$

Using the residue theorem:

$$P_{11}(t) = \Sigma \text{ Residues}$$

$$= \sum_{k=1}^{\alpha+1} \lim_{s \to r_k} \frac{(s-r_k)(s+\beta)^\alpha e^{st}}{s(s-r_1)(s-r_2)\cdots(s-r_k)\cdots(s-r_\alpha)}$$

(3.4)

$$P_{11}(t) = \frac{\beta}{\beta+\alpha\lambda} + \sum_{k=1}^{\alpha} \frac{(r_k+\beta)^\alpha e^{r_k t}}{\prod_{k=1}^{\alpha} r_k(r_k-r_\ell)}$$

$$\ell = 1,2,\cdots,\alpha \quad \ell \neq k .$$

Denoting

$$(r_k+\beta)^\alpha = v_k + w_k i$$

$$\prod_{k=1}^{\alpha} r_k(r_k-r_\ell) = x_k + y_k i \quad \ell \neq k$$

then

(3.5)

$$P_{11}(t) = \frac{\beta}{\beta+\alpha\lambda} + \sum_k((v_k x_k + w_k y_k)\cos b_k t + (v_k y_k - w_k x_k)\cdot$$

(3.5)Cont'd.

$$\sin b_k t)\ \frac{e^{a_k t}}{x_k^2+y_k^2}$$

$$k = 1,3,5,\cdots,\alpha-1.$$

Incidentially, for a bounded solution to exist the real part of all roots must be negative.

The numerical solution is contingent on finding the roots of $s\hat{q}(s)$. This can be done algebraically for $\alpha \leq 4$. The method used is outlined in Appendix C. In addition, for $\alpha = 2$ the inversion of (3.5) in explicit form is shown in Appendix D to illustrate the method.

For a polynomial of unrestricted degree the computer program described in Section V was written.

If a computer is not available, approximating formula $P_{11}(t)$ are given in Section IV.

For $E_{11}(T)$, equation (2.9), the double root at $s = 0$ leads to:

$$E_{11}(T) = \lim_{s\to 0} \frac{d}{ds}\ \frac{s^2(s+\beta)^\alpha\ e^{sT}}{s^2(s-r_1)(s-r_2)\cdots(s-r_\alpha)}$$

$$+ \sum_{k=1}^{\alpha}\ \lim_{s\to r_k} \frac{(s-r_k)(s+\beta)^\alpha e^{sT}}{\prod_{\ell=1}^{\alpha} s^2(s-r_\ell)}$$

which upon derivation and letting $s \to 0$ lead to

(3.6)

$$E_{11}(T) = -\frac{\lambda\alpha(\beta+\frac{1}{2}\lambda(\alpha-1))}{(\beta+\alpha\lambda)^2} + \frac{\lambda\beta}{\beta+\alpha\lambda}T$$

$$+\sum_{k=1}^{\alpha} \lim_{s\to r_k} \frac{(s-r_k)(s+\beta)^{\alpha} e^{sT}}{s^2(s-r_1)(s-r_2)\cdots(s-r_k)\cdots(s-r_{\alpha})}$$

A similar procedure as outlined for $P_{11}(t)$ is used for the last term (modified by multiplication by s).

IV. APPROXIMATIONS AND BOUNDS

A. GENERAL DOWN DISTRIBUTION

Consider

$$\hat{P}_{11}(s) = \frac{1-\hat{f}(s)}{s(1-\hat{f}(s)\cdot\hat{g}(s))}$$

where uptime X ~ exponential, downtime Y ~ G, unspecified.

Then,

$$\hat{P}_{11}(s) = \frac{1-\frac{\lambda}{\lambda+s}}{s\left(1-\frac{\lambda}{\lambda+s}\,\hat{g}(s)\right)}$$

(4.1)

$$\hat{P}_{11}(s) = \frac{1}{s\left(1-\lambda\,\frac{1-\hat{g}(s)}{s}\right)} \quad .$$

By division:

(4.2)

$$\hat{P}(s) = \frac{1}{s}\left(1-\lambda\,\frac{1-\hat{g}(s)}{s} + \lambda^2\left(\frac{1-\hat{g}(s)}{s}\right)^2 - \lambda^3\left(\frac{1-\hat{g}(s)}{s}\right)^3 + \cdots\right)$$

Observing that $1-\hat{g}(s)/s^2$ is the Laplace transform of the equilibrium distribution of G (Reference 4) except for a factor $1/E[Y]$, then

(4.3)

$$\hat{P}_{11}(s) = 1 - R\hat{G}_e(s) + R^2(\hat{G}_e(s))^2 - R^3(\hat{G}_e(s))^3 + \cdots$$

where

$$R = \frac{E[Y]}{E[X]} \;;\; \hat{G}_e(s) = \frac{1-\hat{g}(s)}{E[Y]\cdot s^2}$$

With $R < 1$, convergence is ensured. This implies that $P_{11}(t) \to \frac{1}{2}$ as $t \to \infty$ (steady state result). Inverting (4.3):

$$P_{11}(t) = 1 - RG_e(t) + R^2G_e^{(2)}(t) - R^3G_e^{(3)}(t) + \cdots$$

where $G^{(i)}(t)$ denotes the ith fold convolution of G with itself.

Letting $t \to \infty$,

$$\lim_{t\to\infty} P_{11}(t) = 1 - R + R^2 - R^3 + \cdots = \frac{E[X]}{E[X]+E[Y]}$$

one obtains the steady state result.

B. BOUNDS AND PRECISION

It is a reasonable proposition to try to invert the first term in (4.3). Inversion of the other terms may be difficult unless G_e is a simple distribution.

Even though (4.3) is not a series in the time domain, by probabilistic considerations the effect of truncation may be deduced.

Noting that $R^i\ G_e^{(i)}(t) > R^{i+1}\ G_e^{(i+1)}(t)$, then the approximation of $P_{11}(t)$ will give a lower or upper bound on $P_{11}(t)$ depending on whether the last term is negative or positive, e.g.:

(4.4)

$$P_{11}(t) > 1 - R\cdot G_e(t)$$

(4.5)

$$P_{11}(t) < 1 - R\cdot G_e(t) + R^2G_e^{(2)}(t).$$

The precision with which the bounds are established can be obtained.

Suppose one truncates after one term. Then the maximum value of the remainder is less than R^2. Thus,

(4.6)

$$1 - RG_e(t) + R^2 > P_{11}(t) > 1 - RG_e(t).$$

Having found the lower bound for $P_{11}(t)$, then this value is accurate to within R^2. In a rather typical case with $R = 1/10$ (i.e., steady state $\doteq 0.909$), the lower bound is established to within one per cent.

If it is feasible to compute the upper bound by taking two terms in the expansion, then - by similar argument - the precision of the upper bound will be R^3, i.e., to within 1/10 per cent in the case considered. With R relatively large, the precision is poor. If feasible, more terms must be computed for a desired precision.

C. GAMMA DOWN-DISTRIBUTION

Consider case with $Y \sim \text{Gamma}(\alpha,\beta)$ $\alpha = 1,2,3,\cdots$:

(4.7)

$$R\cdot\hat{G}_e(s) = \frac{\alpha\cdot\lambda}{\beta}\cdot\frac{1-\left(\frac{\beta}{\beta+s}\right)^\alpha}{s^2\left(\frac{\alpha}{\beta}\right)} = \lambda\frac{(\beta+s)^\alpha-\beta^\alpha}{s^2(\beta+s)^\alpha} .$$

Noting that $s = 0$ is root, one may simplify (4.7) as follows:

$$\frac{(\beta+s)^\alpha-\beta^\alpha}{s^2(\beta+s)^\alpha} = \frac{\beta^\alpha+\beta^{\alpha-1}s+\beta^{\alpha-2}(\beta+s)s+\beta^{\alpha-3}(\beta+s)^2s+\cdots+(\beta+s)^{\alpha-1}s-\beta^\alpha}{s^2(\beta+s)^\alpha}$$

$$= \frac{1}{s}\,\frac{1}{\beta}\left\{\left(\frac{\beta}{\beta+s}\right)^{\alpha} + \left(\frac{\beta}{\beta+s}\right)^{\alpha-1} + \cdots + \frac{\beta}{\beta+s}\right\}$$

(4.8)

$$\hat{RG}_e(s) = \frac{1}{s}\,\frac{\lambda}{\beta}\sum_{i}^{\alpha}\left(\frac{\beta}{\beta+s}\right)^{i} .$$

Substituting into (4.3):

(4.9)

$$\hat{P}_{11}(s) = \frac{1}{s}\left[1 - \frac{\lambda}{\beta}\left\{\sum_{i}^{\alpha}\left(\frac{\beta}{\beta+s}\right)^{i}\right\} + \left(\frac{\lambda}{\beta}\right)^{2}\left\{\sum_{i}^{\alpha}\left(\frac{\beta}{\beta+s}\right)^{i}\right\}^{2} - \cdots\right] .$$

Each term within the bracket contains gamma transforms, making the inversion simple.

As an example, consider case with $\alpha = 2$. Then (4.9) becomes:

(4.10)

$$s \cdot \hat{P}_{11}(s) = 1 - \frac{\lambda}{\beta}\left(\frac{\beta}{\beta+s} + \left(\frac{\beta}{\beta+s}\right)^{2}\right) + \left(\frac{\lambda}{\beta}\right)^{2}\left(\left(\frac{\beta}{\beta+s}\right)^{2} + 2\left(\frac{\beta}{\beta+s}\right)^{3} + \left(\frac{\beta}{\beta+s}\right)^{4}\right) \cdots .$$

The resulting inverse for first approximation:

(4.11)

$$\hat{P}_{11}(t) > 1 - \frac{\lambda}{\beta}\left(1 - e^{-\beta t} + 1 - e^{-\beta t}(\beta t+1)\right).$$

Using the correspondence between poisson and gamma distributions, the value can be obtained from cumulative poisson tables.

D. MIXTURE OF GAMMA DOWN-DISTRIBUTION

Consider a mixture of two gamma-distributions with parameters (K,β), (ℓ,γ) and weighted $(1-m)$ and m respectively. The first term in the expansion becomes:

(4.12)

$$\hat{P}_{11}(s) \sim \frac{1}{s}\left[\,1 - \lambda\left(\frac{1-m}{\beta}\,\hat{H}_1(s) + \frac{m}{\gamma}\,\hat{H}_2(s)\right)\right]$$

where

$$\hat{H}_1(s) = \frac{\beta}{\beta+s} + \left(\frac{\beta}{\beta+s}\right)^2 + \cdots + \left(\frac{\beta}{\beta+s}\right)^K$$

$$\hat{H}_2(s) = \frac{\gamma}{\gamma+s} + \left(\frac{\gamma}{\gamma+s}\right)^2 + \cdots + \left(\frac{\gamma}{\gamma+s}\right)^\ell .$$

The first term is seen to be the linear combination of the two terms in the down-distribution. For case $K = 2$, $\ell = 3$, the inversion is:

(4.13)

$$\hat{P}_{11}(t) > 1 - \lambda\left(\frac{1-m}{\beta}\;H_1(t) + \frac{m}{\gamma}\,H_2(t)\right)$$

where

$$H_1(t) = 1 - e^{-\beta t} + 1 - e^{-\beta t}\,(\beta t+1)$$

$$H_2(t) = 1 - e^{-\gamma t} + 1 - e^{-\gamma t}\,(\gamma t+1) + 1 - e^{-\gamma t}\cdot \left(\frac{(\gamma t)^2}{2} + \gamma t + 1\right).$$

In case more terms are required, the inversion will be somewhat complicated in that there will appear cross-terms between the two distributions. The precision of (4.13) will be

$$\lambda^2\left(\frac{(1-m)^2}{\beta^2} + \frac{m^2}{\gamma^2}\right).$$

E. WEIBULL-DISTRIBUTION; SPECIAL CASE

A lower bound can be obtained when downtime is distributed Weibull (β,2). The reliability function is:

(4.14)

$$G(t) = e^{-\beta t^2}, \quad \beta > 0.$$

The expected value is:

$$E[T] = \sqrt{\frac{\pi}{\beta}} \quad .$$

The equilibrium distribution:

$$G_e(t) = \frac{\int_0^t e^{-\beta x^2}\, dx}{E[T]}$$

which becomes upon re-arranging:

$$G_e(t) = \sqrt{\frac{\beta}{\pi}} \left[\frac{1}{\sqrt{2\pi}\sqrt{\frac{1}{2\beta}}} \int_0^t e^{-\frac{1}{2}x^2(2\beta)}\, dx \right] \sqrt{2\pi}\sqrt{\frac{1}{2\beta}}$$

(4.15)

$$G_e(t) = \sqrt{\frac{\beta}{\pi}} \int_0^t e^{-\frac{1}{2}x^2(2\beta)}\, dx.$$

Thus, $G_e(t)$ is ~ normal (0,1/2β). The lower bound becomes:

(4.16)

$$P_{11}(t) > 1 - \lambda\sqrt{\frac{\pi}{\beta}}\left[Z\left(\frac{t}{\sqrt{2\beta}}\right) - \frac{1}{2}\right]$$

where $Z(t/\sqrt{2\beta})$ is normal (0,1). Thus standard normal tables can be used for evaluation. The corresponding precision is $\lambda^2 \cdot \pi/\beta$.

1. Weibull-Distribution; Decreasing Failure Rate

For the special case that the Weibull distribution has shape - parameter $K = 1/p$, $p = 1,2,3,\cdots$, the lower bound with one term in expansion can be found. The equilibrium distribution:

(4.17)

$$G_e(t) = \frac{\beta^p}{p!} \int_0^t e^{-\beta z^{\frac{1}{p}}} dz$$

which upon substitution becomes:

(4.18)

$$G_e(t) = \int_0^{t^{\frac{1}{p}}} \frac{\beta^p}{(p-1)!} e^{-\beta x} x^{p-1} dx$$

where

$$z = x^p .$$

Equation (4.18) is the gamma-distribution in $x(=z^{\frac{1}{p}})$ with parameters (p,β). The lower bound becomes:

$$P_{11}(t) > 1 - \lambda \frac{p!}{\beta^p} \Gamma(t;p,\beta)$$

with precision:

$$\left[\frac{p!}{\beta^p} \lambda \right]^2 .$$

As an example, let $k = \frac{1}{2}$, i.e., $p = 2$. Thus,

$$P_{11}(t) > 1 - \lambda \frac{2}{\beta^2} \left[1 - e^{-\beta\sqrt{t}} \left(\beta\sqrt{t} + 1\right)\right] .$$

The value of term in brackets can be obtained from poisson tables.

V. COMPUTER PROGRAM

The computer program was written in FORTRAN IV and run on the IBM system/360 model 67 computer at the U. S. Naval Postgraduate School. The program listing is shown in Appendix E.

The main sequence in the program is:

(i) Input data: α, β, λ, time increments and number of time steps (named respectively M, BETA, UPRATE, TSTEP, L in program).

(ii) Generate coefficients of polynomial in denominator of (2.7).

(iii) Call library subroutine DPOLRT to solve for roots, using Newton-Raphson's method.

(iv) Compute and store coefficients for cosine, sine terms in $P_{11}(t)$, $P_{01}(t)$, $M_{11}(t)$ and $E_{11}(t)$; e.g., see (3.5) and (3.6)

(v) Sum cosine coefficients for $P_{11}(t)$ and add to steady state value of $P_{11}(t)$, to give an indication of accuracy of final result. Theoretically the sum should add to 1. Flagging value IR provided by subroutine to indicate accuracy of root extraction should read zero.

(vi) Enter DO-loop for incrementing time and for final calculations.

(vii) Print results: t, $P_{11}(t)$, $P_{01}(t)$, $M_{11}(t)$, $E_{11}(t)$, named: T, PROBUP, DPRBUP, RNWAL, EXPUP respectively in the program.

Sample output is shown in Appendix F.

APPENDIX A: NATURE OF ROOTS

Polynomial considered:

(A.1)

$$P(x) = x^{\alpha} + \lambda(x^{\alpha-1} + \beta x^{\alpha-2} + \cdots + \beta^{\alpha-1}) \; ; \; \beta > \lambda \; .$$

Multiply both sides by $x-\beta$, to obtain:

(A.2)

$$Q(x) = x^{\alpha+1} - (\beta-\lambda)x^{\alpha} - \lambda\beta^{\alpha} = P(x)(x-\beta).$$

Let α be even.

Then by Descartes' Rule of Sign [3], with only one variation of sign, there exist at most one real, positive root (clearly $x = \beta$).

Consider

(A.3)

$$Q(-x) = -|x|^{\alpha+1} - (\beta-\lambda)x^{\alpha} - \lambda\beta^{\alpha}.$$

Since there are no variations in sign, no negative real roots exist. CONCLUSION: For α even, (A.1) can only have complex roots.

Let α be odd.

By (A.2), one variation indicates the positive root $x = \beta$. Consider

(A.4)

$$Q(-x) = x^{\alpha+1} + (\beta-\lambda)|x^{\alpha}| - \lambda\beta^{\alpha}.$$

One variation imply one real, negative root. CONCLUSION: For α odd, (A.1) has one negative, real root, $(\alpha-1)$ complex roots.

The roots of (A.1) are of the form

$$r_j = a_j + b_j i \quad ; \; j = 1,2,\cdots\alpha \text{ with } b_1 = 0 \text{ if } \alpha \text{ is odd.}$$

Since $x = s+\beta$, the roots of s will be

$$r_j' = r_j - \beta = a_j - \beta + b_j i$$

and

$$\sum_{j=1}^{\alpha} r_j' = -\lambda - \alpha\beta.$$

That all $a_j < 0$ can be verified by Hurwitz Criterion (3). However, since $E[X+Y] < \infty$ and $F{*}G(t)$ is non-lattice, the system will be stable, implying that $a_j < 0$ for all j [4] (* denotes convolution).

APPENDIX B: NON-EXISTENCE OF MULTIPLE ROOTS

Polynomial considered:

(B.1)

$$P(x) = x^{\alpha} + \lambda\,(x^{\alpha-1} + \beta x^{\alpha-2} + \cdots + \beta^{\alpha-1});\ \beta > \lambda.$$

For simplicity, let $\alpha = 6$, and assume that a double and a single complex root exist:

$$r_{1,2} = a \pm bi$$

$$r_{3,4} = a \pm bi$$

$$r_{5,6} = c \pm di.$$

Consider the usual relationship between roots and coefficients [3]:

(B.2)

$$4a + 2c = -\lambda$$; sum one root at a time

(B.3)

$$(a^2+b^2)^2(c^2+d^2) = \lambda\beta^5$$; sum 6 at a time (i.e., product of roots)

(B.4)

$$2(a^2+b^2)+(c^2+d^2)+4a^2+8ac = \beta\lambda$$; sum roots two at a time.

Using Cauchy's inequality on (B.3):

(B.5)

$$\frac{2(a^2+b^2)+(c^2+d^2)}{3} > \sqrt[3]{(a^2+b^2)^2(c^2+d^2)} = \sqrt[3]{\lambda\beta^5}$$

By (B.4), then

(B.6)

$$2(a^2+b^2)+c^2+d^2 = \beta\lambda - (4a^2+8ac)$$

$$= \beta\lambda - 4((a+c)^2-c^2) < \beta\lambda$$

since $(a+c)^2-c^2 > 0$.

Equations (B.5) and (B.6) are in contradiction, since $3\sqrt[3]{\lambda\beta^5} > \beta\lambda$ for all values of $\beta > \lambda$. Thus double roots are not possible. The proof can be extended to any α and any multiplicity of complex roots considered. The proof holds also for α odd.

The two inequalities established for $\alpha = 6$ hold in general since all terms of form (a^2+b^2) will appear in isolation from all possible cross terms of form $(a\cdot c)$.

APPENDIX C: POLYNOMIAL DEGREE 3 AND 4

Provided $\beta > \lambda$, the roots of polynomial of degree 3 and 4 can easily be found [5].

$\alpha = 3$

$$P(y) = y^3 + \lambda y^2 + \lambda\beta y + \lambda\beta^2 \;;\; y = s+\beta.$$

Denote:

$$a = \frac{1}{3}(3\lambda\beta - \lambda^2)$$

$$b = \frac{1}{27}(2\lambda^3 - 9\lambda^2\beta + 27\lambda\beta^2)$$

$$A = \left(-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right)^{\frac{1}{3}}$$

$$B = \left(-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right)^{\frac{1}{3}} .$$

Then the roots, solving for s, are:

$$r_1 = A + B - \beta - \frac{\lambda}{3} \qquad \text{(real root)}$$

$$r_{2,3} = -\frac{1}{2}(A+B) \pm \frac{1}{2}(A-B)\sqrt{3}\, i - \beta - \frac{\lambda}{3} \qquad \text{(complex)}$$

$\alpha = 4$

$$P(x) = x^4 + \lambda x^3 + \lambda\beta x^2 + \lambda\beta^2 x + \lambda\beta^3 \;;\; x = s+\beta.$$

Denote:

$$a = \frac{2}{3}(\lambda - 6\beta)\lambda\beta^2$$

$$b = \frac{5}{27}(9\beta - 4\lambda)\lambda^2\beta^3$$

$$m = 2\left(-\frac{a}{3}\right)^{\frac{1}{2}}$$

$$3\theta = \arccos\left(\frac{3b}{a\cdot m}\right)$$

$$y = m\cos\theta + \lambda\beta/3$$

$$R = \left(\frac{\lambda^2}{4} - \lambda\beta + y\right)^{\frac{1}{2}}$$

$$D_{1,2} = \left|\frac{1}{2}\lambda^2 - \lambda\beta - y \pm (4\lambda^2\beta - 8\lambda\beta^2 - \lambda^3)/4R\right|^{\frac{1}{2}}.$$

Then, the four complex roots, solving for s, are:

$$r_{1,2} = -\frac{1}{2}R - \left(\frac{1}{4}\lambda+\beta\right) \pm \frac{1}{2}D_2 i$$

$$r_{3,4} = \frac{1}{2}R - \left(\frac{1}{4}\lambda+\beta\right) \pm \frac{1}{2}D_1 i$$

The simplicity in form of polynomial

$$P(x) = x^{\alpha+1} - (\beta-\lambda)x^{\alpha} - \lambda\beta^{\alpha}$$

make solving for roots for $\alpha > 4$ by manual methods easy, for example, by Graeffe's Method [3].

APPENDIX D: AVAILABILITY FUNCTION IN EXPLICIT FORM; $\alpha = 2$

In order to illustrate the inversion method used, consider $P_{11}(t)$ for $\alpha = 2$:

(D.1)

$$P_{11}(t) = L^{-1}\left\{ \frac{(s+\beta)^2}{s\left((s+\beta)^2+\lambda(s+\beta)+\lambda\beta\right)} \right\} = L^{-1}\left\{ \frac{(s+\beta)^2}{s(s-r_1)(s-r_2)} \right\}$$

where

$$r_{1,2} = \frac{1}{2}\left(-\lambda-2\beta\pm\sqrt{4\lambda\beta-\lambda^2}\, i\right) \quad ; \; \beta > \lambda$$

$$= \frac{1}{2}\left(-\lambda-2\beta\pm Ai\right) \quad ; \; A = \sqrt{4\lambda\beta-\lambda^2} \; .$$

Then,

$$P_{11}(t) = \lim_{s\to 0} \frac{s(s+\beta)^2\, e^{st}}{s(s-r_1)(s-r_2)} + \lim_{s\to r_1} \frac{(s-r_1)(s+\beta)^2\, e^{st}}{s(s-r_1)(s-r_2)}$$

$$+ \lim_{s\to r_2} \frac{(s-r_2)(s+\beta)^2\, e^{st}}{s(s-r_1)(s-r_2)}$$

$$= \frac{\beta^2}{r_1 r_2} + \frac{(r_1+\beta)^2}{r_1(r_1-r_2)}\, e^{r_1 t} + \frac{(r_2+\beta)^2}{r_2(r_2-r_1)}\, e^{r_2 t} \; .$$

Since the last two terms are complex conjugate numbers, they may be collected and expressed in terms of trigonometric functions:

$$P_{11}(t) = \frac{\beta^2}{\beta^2+2\lambda\beta} + e^{-(\beta+\frac{\lambda}{2})t} \cdot \frac{(r_1+\beta)^2 r_2(r_2-r_1)e^{\frac{A}{2}t} + (r_2+\beta)^2 r_1(r_1-r_2)e^{-\frac{A}{2}t}}{r_1 r_2(r_1-r_2)(r_2-r_1)}$$

$$= \frac{\beta}{\beta+2\lambda} + 2e^{-(\beta+\frac{\lambda}{2})t} \frac{(ac+db)\cos\frac{A}{2}t + (ad-bc)\sin\frac{A}{2}t}{c^2+d^2}$$

where

$$(r_1+\beta)^2 = a + bi \qquad ; \quad (r_2+\beta)^2 = a - bi$$

$$r_1(r_1-r_2) = c + di \qquad ; \quad r_2(r_2- r_1) = c - di.$$

Substituting in actual values, the result is:

$$P_{11}(t) = \frac{\beta}{\beta+2\lambda} + \frac{2\lambda}{A(\beta+2\lambda)} \left\{ A \cos \frac{A}{2}t + (\beta-\lambda) \sin \frac{A}{2}t \right\} e^{-(\beta+\frac{\lambda}{2})t}$$

Closed form solutions were also found for:

$$M_{11}(t) = - \frac{\lambda(2\beta+\lambda)}{(\beta+2\lambda)^2} + \frac{\lambda\beta}{\beta+2\lambda} t + \frac{\lambda e^{-(\beta+\frac{\lambda}{2})t}}{(\beta+2\lambda)^2 A} \left\{ (2\beta^2+\lambda^2)\sin \frac{A}{2}t \right.$$

$$\left. + A\cdot(2\beta+\lambda)\cos \frac{A}{2}t \right\} .$$

$$P_{01}(t) = \frac{\beta}{\beta+2\lambda} - \frac{\beta}{(\beta+2\lambda)A} \left\{ (2\beta+\lambda)\sin \frac{A}{2} + A \cos \frac{A}{2}t \right\} e^{-(\beta+\frac{\lambda}{2})t}$$

APPENDIX E: COMPUTER LISTING

```
      IMPLICIT REAL*8(A-H,O-Y),COMPLEX*16(Z)
      DIMENSION XCOF(100),COF(100),ROOTR(100),ROOTI(100)
      DIMENSION ZR(100),ZROOT(100),ZNUM(100),ZDNM(100)
      DIMENSION SCOEF(100),QCOEF(100)
      DIMENSION QEFF(100)
      DIMENSION DQCOEF(100),DSCOEF(100),DQEFF(100)
      DIMENSION RQEFF(100),RQCOEF(100),RSCOEF(100)
      DIMENSION EQCOEF(100),ESCOEF(100),EQEFF(100)
      DIMENSION XNUM(2),XDNM(2),XRDNM(2),XR(2)
      EQUIVALENCE (ZZDNM,XDNM),(ZZNUM,XNUM),(ZRDNM,XRDNM),(ZZR,XR)
      M=5
      UPRATE=1.0D0/4.5D0
      BETA=10.0D0
      TSTEP=0.01D0
C            GENERATE POLYNOMIAL AND SOLVE FOR ROOTS
      IUP=0
      J=M+1
      XCOF(J)=1.0D0
      XCOF(M)=UPRATE
      SUM1=0.0D0
      DO 1000 N=2,M
      IUP=IUP+1
      L=M-N+1
      XCOF(L)=UPRATE*BETA**IUP
 1000 CONTINUE
C            NOW FOR INVERSION OF LAPLACE TRANSFORM
      CALL DPOLRT(XCOF,COF,M,ROOTR,ROOTI,IER)
      DO 1500 I=1,M
      ZROOT(I)=DCMPLX(ROOTR(I),ROOTI(I))
      ZR(I)=ZROOT(I)-BETA
      ZNUM(I)=ZROOT(I)**M
 1500 CONTINUE
C            ABOVE DID CALCULATE NUMERATORS.  NOW FOR DENUMERATORS
      DO 2000 I=1,M
      IF(I.NE.1) GO TO 2001
      ZPROD=ZR(1)-ZR(2)
      IF(M.GT.2) GO TO 2002
      ZDNM(1)=ZPROD*ZR(1)
      GO TO 3502
 2001 ZPROD=ZR(I)-ZR(1)
      IPROD=2
      IF(M.EQ.2) GO TO 2501
      GO TO 2003
 2002 DYMMY=0.0
      IPROD=3
 2003 DUM=0.0
      DO 2500 J=IPROD,M
      IF((J.EQ.I).AND.(J.EQ.M)) GO TO 2501
```

```
      IF(J.EQ.I) GO TO 2500
      ZPROD=ZPROD*(ZR(I)-ZR(J))
 2500 CONTINUE
 2501 ZDNM(I)=ZPROD*ZR(I)
 2000 CONTINUE
C              NEXT FOLLOWS SINE AND COSINE COEFFICIENTS
      ICHECK=(M/2)*2-M
      IF(ICHECK) 3501,3502,3501
 3502 IMAG=1
      GO TO 3503
 3501 IMAG=2
 3503 DUMMY=1.0
      ILIM=M-1
      DO 3500 I=IMAG,ILIM,2
      ZZDNM=ZDNM(I)
      ZZNUM=ZNUM(I)
      ZRDNM=ZDNM(I)*ZR(I)
      QNUM=XNUM(1)*XDNM(1)+XNUM(2)*XDNM(2)
      QDNM=XDNM(1)**2+XDNM(2)**2
      QCOEF(I)=2.0D0*QNUM/QDNM
      SNUM=XNUM(1)*XDNM(2)-XNUM(2)*XDNM(1)
      SCOEF(I)=2.0D0*SNUM/QDNM
      DQCOEFF(I)=2.0D0*BETA**M*XDNM(1)/QDNM
      DSCOEFF(I)=2.0D0*BETA**M*XDNM(2)/QDNM
      RDNM=XRDNM(1)**2+XRDNM(2)**2
      RQCOEFF(I)=2.0D0*UPRATE*BETA**M*XRDNM(1)/RDNM
      RSCOEFF(I)=2.0D0*UPRATE*BETA**M*XRDNM(2)/RDNM
      EQNUM=XNUM(1)*XRDNM(1)+XNUM(2)*XRDNM(2)
      ESNUM=XNUM(1)*XRDNM(2)-XNUM(2)*XRDNM(1)
      ESCOEFF(I)=2.0D0*ESNUM/RDNM
      EQCOEFF(I)=2.0D0*EQNUM/RDNM
      SUM1=SUM1+QCOEF(I)
 3500 CONTINUE
C              NOW FOR SINGLE REAL COEFFICIENTS
      IF(IMAG.NE.2) GO TO 3504
      ZZDNM=ZDNM(1)
      ZZNUM=ZNUM(1)
      ZRDNM=ZDNM(1)*ZR(1)
      QEFF(1)=XNUM(1)/XDNM(1)
      DQEFF(1)=BETA**M/XDNM(1)
      RQEFF(1)=UPRATE*BETA**M/XRDNM(1)
      EQEFF(1)=XNUM(1)/XRDNM(1)
      SUM1=SUM1+QEFF(1)
C              PRELIMINARIES TO LOOPING FOR TIME
 3504 DUMMY=0.0
      RMND1=BETA/(BETA+DFLOAT(M)*UPRATE)
      ACURCY=RMND1+SUM1
      WRITE(6,1)
```

```
1     FORMAT('1',10X,'CHECK FOR ACCURACY OF NUMERICAL SOLUTION'///)
      WRITE(6,2) ACURCY,IER
2     FORMAT(' ',10X,'INDICATOR FOR ACCURACY=',F16.12,//,11X,'SUBROUTINE
     1 FLAGGING VALUE=',I13/////)
      WRITE(6,3)
3     FORMAT(' ',11X,'TIME',3X,'AVAILABILITY',5X,'AVAIL/DOWN',6X,'E(RENE
     1WAL)',7X,'E(TIME ON)'//)
      RCNST=UPRATE*BETA/(BETA+DFLOAT(M)*UPRATE)
      RCONST=UPRATE*DFLOAT(M)*(BETA+0.5D0*UPRATE*(DFLOAT(M)-1.0D0))/
     1(BETA+DFLOAT(M)*UPRATE)**2
      ECNST=UPRATE*DFLOAT(M)*(DFLOAT(M)+1.0D0)/(2.0D0*(BETA+UPRATE*DFLO
     1AT(M))**2)
      DO 4500 L=1,100
      T=TSTEP*DFLOAT(L)
      RMND=0.0D0
      TERM=0.0D0
      DRMND=0.0D0
      DTERM=0.0D0
      RRMND=0.0D0
      RTERM=0.0D0
      ERMND=0.0D0
      ETERM=0.0D0
      DO 5000 I=IMAG,ILIM,2
      ZZR=ZR(I)
      QOS=DCOS(XR(2)*T)
      SIN=DSIN(XR(2)*T)
      EXPON=DEXP(XR(1)*T)
      RMND=RMND+EXPON*(QOS*OCOEF(1)+SIN*SCOEF(I))
      DRMND=DRMND+EXPON*(QOS*DOCOEF(I)+SIN*DSCOEF(I))
      RRMND=RRMND+EXPON*(QOS*ROCOEF(I)+SIN*RSCOEF(I))
      ERMND=ERMND+EXPON*(QOS*EOCOEF(I)+SIN*ESCOEF(I))
5000  CONTINUE
      IF(IMAG.NE.2) GO TO 5001
      REXPON=DEXP((ROOTR(1)-BETA)*T)
      TERM=OEFF(1)*REXPON
      DTERM=DOEFF(1)*REXPON
      RTERM=ROEFF(1)*REXPON
      ETERM=EOEFF(1)*REXPON
5001  PROBUP=RMND1+RMND+TERM
      DPRBUP=RMND1+DRMND+DTERM
      RNWAL=-RCONST+RCNST*T+RRMND+RTERM
      EXPUP=ECNST+RMND1*T+ERMND+ETERM
      WRITE(6,5003) T,PROBUP,DPRBUP,RNWAL,EXPUP
5003  FORMAT(' ',10X,F6.3,4X,F8.6,8X,F8.6,7X,F9.6,9X,F8.6)
4500  CONTINUE
      STOP
      END
```

APPENDIX F: SAMPLE RESULTS

```
CHECK FOR ACCURACY OF NUMERICAL SOLUTION:

 INDICATOR FOR ACCURACY=  1.000000000000
 SUBROUTINE FLAGGING VALUE=             0

 TIME    AVAILABILITY        AVAIL/DOWN        E(RENEWAL)            E(TIME ON)

 0.010     0.997780          0.000000          0.000000              0.009989
 0.020     0.995565          0.000002          0.000000              0.019956
 0.030     0.993356          0.000016          0.000000              0.029900
 0.040     0.991151          0.000061          0.000000              0.039823
 0.050     0.988951          0.000172          0.000000              0.049723
 0.060     0.986756          0.000394          0.000001              0.059602
 0.070     0.984567          0.000783          0.000002              0.069458
 0.080     0.982384          0.001407          0.000005              0.079293
 0.090     0.980207          0.002335          0.000009              0.089106
 0.100     0.978038          0.003645          0.000015              0.098897
 0.110     0.975877          0.005410          0.000025              0.108667
 0.120     0.973725          0.007706          0.000040              0.118415
 0.130     0.971584          0.010603          0.000060              0.128141
 0.140     0.969455          0.014166          0.000087              0.137847
 0.150     0.967339          0.018453          0.000123              0.147531
 0.160     0.965238          0.023512          0.000170              0.157193
 0.170     0.963154          0.029386          0.000229              0.166835
 0.180     0.961089          0.036106          0.000301              0.176457
 0.190     0.959044          0.043692          0.000390              0.186057
 0.200     0.957021          0.052157          0.000496              0.195638
 0.210     0.955023          0.061504          0.000622              0.205198
 0.220     0.953050          0.071727          0.000770              0.214738
 0.230     0.951106          0.082810          0.000942              0.224259
 0.240     0.949192          0.094731          0.001139              0.233760
 0.250     0.947309          0.107461          0.001363              0.243243
 0.260     0.945460          0.120962          0.001617              0.252707
 0.270     0.943645          0.135194          0.001901              0.262152
 0.280     0.941867          0.150110          0.002218              0.271580
 0.290     0.940127          0.165660          0.002569              0.280990
 0.300     0.938425          0.181790          0.002955              0.290382
 0.310     0.936764          0.198444          0.003377              0.299758
 0.320     0.935144          0.215565          0.003837              0.309118
 0.330     0.933566          0.233093          0.004336              0.318461
 0.340     0.932031          0.250969          0.004873              0.327789
 0.350     0.930540          0.269135          0.005451              0.337102
 0.360     0.929092          0.287531          0.006070              0.346400
 0.370     0.927688          0.306100          0.006729              0.355684
 0.380     0.926329          0.324787          0.007430              0.364954
 0.390     0.925015          0.343536          0.008173              0.374211
 0.400     0.923745          0.362297          0.008957              0.383454
 0.410     0.922520          0.381019          0.009783              0.392686
 0.420     0.921338          0.399655          0.010650              0.401905
 0.430     0.920201          0.418162          0.011559              0.411113
 0.440     0.919107          0.436497          0.012509              0.420309
 0.450     0.918056          0.454622          0.013499              0.429495
 0.460     0.917047          0.472502          0.014529              0.438670
 0.470     0.916080          0.490103          0.015599              0.447836
 0.480     0.915153          0.507396          0.016707              0.456992
 0.490     0.914267          0.524355          0.017854              0.466139
 0.500     0.913420          0.540956          0.019037              0.475278
```

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13. ABSTRACT

This paper considers an alternating renewal process with time to failure being distributed exponential (λ), and with a general distribution for time to repair. Bounds and the precision of the bounds for the availability function were obtained.

A computer program was written to solve for the availability function and some other quantities when the down distribution was a gamma (α,β), α integer. The availability function with down distribution being a mixture of gamma-functions is considered.

KEY WORDS	LINK A		LINK B		LINK C	
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Exponential Up-Distribution						
General Down-Distribution						
Bounds						
Gamma Down-Distribution						

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